

# Homework 6

## OR 41

For each numerical answer, you need to provide a concise explanation why it is true. E.g. it is not enough to write  $\frac{16!}{2!3!11!}$ , but you need to explain how you arrive at this result.

If you give more than one correct way to solve a question, you will get bonus points.

1. ( 10 points) For each of the following sentences, write down a sentence which is the negation of it. The negation sentence should not contain the word “no”, or “not”. We assume that all cows in the world are brown, black, or white. We also assume that Uncle Bob has at least one cow.

The sentences, whose negation you need to provide are:

- Every cow is either brown or has 4 legs.
  - Uncle Bob has a white cow with 3 legs.
2. (10 points) Suppose that every person is equally likely to have been born on day  $1, 2, \dots, 31$  of some month. What is the smallest  $n$  which makes the following statement true:
    - If we choose  $n$  people at random, the probability that no two of them have their birthday on the same day is less than 0.25?
  3. Consider the polygraph problem as discussed in class.
    - (a) Suppose that 12 % of the population is guilty, and the reliability of the test for guilt is 90 %. How reliable should be the test for innocence, so that
$$P(\text{innocent} \mid \text{test says guilty}) \leq 0.01 ?$$
(15 points)
    - (b) Suppose that the reliability of the test for innocence is fixed at 98 %. What is the best, (that is smallest) value of
$$P(\text{innocent} \mid \text{test says guilty})$$
that can be achieved by increasing the reliability of the test for guilt? (15 points)
    - (c) Suppose that

- $p$  percent of the population is guilty,
- the reliability of the test both for guilt and innocence is  $1 - p$ .

Say,  $p = 10$  means that 10 % is guilty, and out of 100 guilty people who are tested, 90 are found guilty; also out of 100 innocent people tested, 90 are found innocent.

Find

$$P(\text{innocent} \mid \text{test says guilty})$$

for general  $p$  between 0 and 1. (Hint: this number will be entirely independent of the actual value of  $p$ .)

The number you get will be something that is at first difficult to believe. Try to construct an argument that will convince even a layman of its correctness, as follows: let  $p = 0.1$ , and consider a population of 100 people. If each one of them is tested for guilt,

- how many will be found guilty, and innocent, respectively?
- out of the ones that were found guilty, how many will be guilty, and innocent, respectively?
- out of the ones that were found innocent, how many will be guilty, and innocent, respectively?

Repeat this exercise with a population of 10,000 people, and  $p = 0.01$ . (30 points)

- Ross 2.30 (15 points)
- Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers? (10 points)
- We know that two events  $A$  and  $B$  are independent if and only if  $P(A \cap B) = P(A)P(B)$ .  
Let  
 $A$  = the event that a card drawn from an ordinary deck is an ace,  
 $B$  = the event that a card drawn from an ordinary deck is a spade.  
Show that  $A$  and  $B$  are independent events. (10 points)
- Ross 3.4 (20 points)
- Ross 3.6 (15 points)
- Ross 3.8 (20 points)
- Consider two boxes, one containing 1 red and 1 green marble, the other containing 2 red and 1 green marble. A box is selected at random, and a marble is drawn at random from the selected box.
  - What is the probability the marble is red?
  - What is the probability the first box was the one selected, given that the marble is green?

(15 points)

11. If 8 rooks (castles) are randomly placed on a chess board, what is the probability that none of the rooks can capture any of the others? Note that a rook can capture another rook if it is in the same row or column. So, we can rephrase the question: What is the probability that no row or column contains more than one rook? (15 points)
12. If two cards are randomly selected from a regular deck, what is the probability they are both diamonds? What is the probability they form a blackjack? (A blackjack is formed by an Ace together with a 10, J, Q, or K)
  - (a) Answer both of these questions using our "old" method (e.g. counting the number of ways to get a blackjack divided by the total number of ways to pick two cards). (10 points)
  - (b) Answer both of these questions using conditioning and confirm that you get the same answers as you did in part (a). (10 points)