Homework 7 solutions
OR 41

1. (Monty Hall with n doors)

   Define the events:

   - A : the contestant wins with the switching strategy.
   - B_1 : the original door has the prize.
   - B_2 : the original door does not have the prize.

   We have

   $P(A|B_1) = 0$ (obvious)

   $P(B_1) = 1/n$ (obvious)

   $P(B_2) = (n-1)/n$ (obvious)

   $P(A|B_2) = 1/(n-2)$. Reason: if the contestant picked a prizeless door originally, then Monty opens *another* prizeless door. So the contestant will switch to one of the remaining $n-2$ doors, one of which does have the prize.

   So

   $$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) \\ &= 0 \cdot \frac{1}{n} + \frac{1}{n-2} \cdot \frac{1}{n-1} \\ &= \frac{n-1}{n(n-2)} \end{aligned}$$

2. The roulette problem.

   Let $p = 18/38$, the probability of getting a red. The possibilities are:

   - Win at first. Winning is 1 \$, probability is p.
   - Lose at first, win at second. Winning is $2-1=1$ \$, probability is $(1-p)p$.
   - Lose at first, lose at second. Winning is -3 \$, probability is $(1-p)^2$.

   The expected winning is

   $$\begin{aligned} 1 \times (p + p(1-p)) - 3 \times (1-p)^2 &= p(2-p) - 3 \times (1-p)^2 \\ &= \frac{18}{38} \times \frac{58}{38} - 3 \times (\frac{20}{38})^2 \\ &= -\frac{156}{38^2} = -0.1080 \end{aligned}$$

The expected winning of just one bet on the red is

$$-\frac{2}{38} = -0.0526$$

so the "winning strategy" is actually worse!

With the ℓ-strategy, the following can happen:

- Win at first. Winning is 1 \$, probability is p. (bet was 1 \$)
- Lose at first, win at second. Winning is $2 - 1 = 1$ \$, probability is $(1-p)p$. (bet was 2 \$)
- Lose at first, lose at second, win at third. Winning is $4 - 2 - 1 = 1$ \$, probability is $(1-p)^2 p$. (bet was 2^2 \$)

$$\vdots$$

- Lose at first, lose at second, …, win at (ℓ+1)st. Winning is $2^\ell - 2^{\ell-1} - \cdots - 1 = 1$ \$, probability is $(1-p)^\ell p$. (bet was 2^ℓ \$)
- Lose at first, lose at second, …, lose at (ℓ+1)st. Winning is $-(2^\ell + 2^{\ell-1} + \cdots + 1) = -(2^{\ell+1} - 1)$ \$, probability is $(1-p)^{\ell+1}$. (bet was 2^ℓ \$)

The expected winning is

$$\begin{aligned} p\left(1 + (1-p) + \cdots + (1-p)^\ell\right) - 2^\ell(1-p)^{\ell+1} &= p\frac{(1-p)^{\ell+1} - 1}{1-p-1} - (2^{\ell+1} - 1)(1-p)^{\ell+1} \\ &= \left(1 - (1-p)^{\ell+1}\right) - (2^{\ell+1} - 1)(1-p)^{\ell+1} \end{aligned}$$

The negative of this is the expected loss, which is equal to

$$(2^{\ell+1} - 1)(1-p)^{\ell+1} - \left(1 - (1-p)^{\ell+1}\right)$$

This is equal to

$$[2(1-p)]^{\ell+1}$$

plus some other stuff, which is either constant, or goes to zero, as $\ell \to \infty$. Since

$$2(1-p) > 1,$$

(this is equivalent to $p < 1/2$, i.e. to the probability of winning in any round being less than half), the expected monotonically loss goes to $+\infty$, as ℓ grows.

3. The craps problem

Denote by B_i the event that the first throw is i, and by A the event that the gambler wins. Then

$$\begin{aligned} P(A) &= \sum_{i=2}^{12} P(A|B_i)P(B_i) \\ &= P(A|B_7)P(B_7) + P(A|B_{11})P(B_{11}) + \sum_{i=4,5,6,8,9,10} P(A|B_i)P(B_i) \\ &= 1 \cdot \frac{6}{36} + 1 \cdot \frac{2}{36} + \sum_{i=4,5,6,8,9,10} P(A|B_i)P(B_i). \end{aligned}$$

It remains to find $P(A|B_i)$ for $i = 4, 5, 6, 8, 9, 10$. For brevity, write

$$p_i = P(B_i),\ q_i = 1 - P(B_7) - P(B_i).$$

Then for $i = 4, 5, 6, 8, 9, 10$

$$\begin{aligned} P(A|B_i) &= p_i + q_i p_i + q_i^2 p_i + q_i^3 p_i + \ldots \\ &= p_i(1 + q_i + q_i^2 + q_i^3 + \ldots) \\ &= \frac{p_i}{1 - q_i} \\ &= \frac{p_i}{p_i + p_7} \end{aligned}$$

The reason is as follows: if our first throw is a point, then in each subsequent throw the probability of getting neither our point i, nor 7 is q_i. To win, we can throw our point in the second throw; throw neither our point, nor 7 in the second throw, then throw our point in the third throw; etc. So using

$$P(B_7) = \frac{6}{36},\ p_4 = p_{10} = \frac{3}{36},\ p_5 = p_9 = \frac{4}{36},\ p_6 = p_8 = \frac{5}{36},\ p_9 = \frac{4}{36},\ p_{10} = \frac{3}{36}$$

we obtain

$$\begin{aligned} P(A|B_4) &= P(A|B_{10}) = \frac{3}{9}, \\ P(A|B_5) &= P(A|B_9) = \frac{4}{10}, \\ P(A|B_6) &= P(A|B_8) = \frac{5}{11}. \end{aligned}$$

Plugging these in, the final result is

$$\begin{aligned} P(A) &= 1 \cdot \frac{6}{36} + 1 \cdot \frac{2}{36} + 2 \cdot \frac{3}{9} \cdot \frac{3}{36} + 2 \cdot \frac{4}{10} \cdot \frac{4}{36} + 2 \cdot \frac{5}{11} \cdot \frac{5}{36} \\ &= 0.49292929 \end{aligned}$$

So, amazing as it is, this bet results in an "almost" fair game – the chance of losing is just slightly smaller than 1/2.

4. (the independents, liberals, etc. problem)

   (a)

$$\begin{aligned} P(\text{I}|\text{Voted}) &= \frac{P(\text{I} \cap \text{Voted})}{P(\text{Voted})} \\ &= \frac{P(\text{Voted}|\text{I}) \cdot P(\text{I})}{P(\text{Voted}|\text{I}) \cdot P(\text{I}) + P(\text{Voted}|\text{L}) \cdot P(\text{L}) + P(\text{Voted}|\text{C}) \cdot P(\text{C})} \\ &= \frac{(0.35) \cdot (0.46)}{(0.35) \cdot (0.46) + (0.62) \cdot (0.30) + (0.58) \cdot (0.24)} \\ &= 0.331 \end{aligned}$$

   (b)

$$P(\text{Voted}) = \text{denominator of answer above} = 0.4862$$

5. (value of raffle ticket problem)

$$\begin{aligned} P(X=98) &= P(X=48) = P(X=23) = 0.005 \\ P(X=8) &= 0.01 \\ P(X-2) &= 0.97 \\ E[X] &= 0.005 \times (98+48+23) + 0.01 \times 8 + 0.975 \times (-2) = -1.025 \end{aligned}$$

6. (the insurance agent problem)

   Let

   $X = $ #policies she sells tomorrow

   $X \sim Bin(4, 0.20)$

   (a)

   $$P(X \geq 3) = P(X=3) + P(X=4) = \binom{4}{3} \cdot (0.2)^3 \cdot (0.8) + \binom{4}{4} \cdot (0.2)^4 = 0.0272$$

   (b)

   $$E[X] = 4 \times 0.2 = 0.8$$

7. (Ross 3.12, the buses and students problem)

   (a) $E[X]$ should be higher since the method we use to select the bus gives more weight to more heavily loaded buses. For $E[Y]$, each bus is equally likely to be chosen.

   (b)

   $$\begin{aligned} E[X] = &\ 39 \cdot (39/152) + 33 \cdot (33/152) + 46 \cdot (46/152) + 34 \cdot (34/152) = 38.697 \\ E[Y] = &\ 39 \cdot (0.25) + 33 \cdot (0.25) + 46 \cdot (0.25) + 34 \cdot (0.25) = 38 \end{aligned}$$

8. (Ross 3.15/a, the tennis player problem)

   Let

   $X = $ number of matches played

   There are only 2 possible values of X: 2 and 3.

   Let us denote the event that match 1 is won by player i, and match 2 is won by player j, by (i, j). Then the probability of all 4 events

   $$\begin{gathered} (1,2) \\ (2,1) \\ (1,1) \\ (2,2) \end{gathered}$$

   is $1/4$. In the first 2 cases, $X = 3$, in the last 2, $X = 2$. So

   $$E[X] = 2 \cdot P(X=2) + 3 \cdot P(X=3) = 2 \cdot (0.5) + 3 \cdot (0.5) = 2.5$$

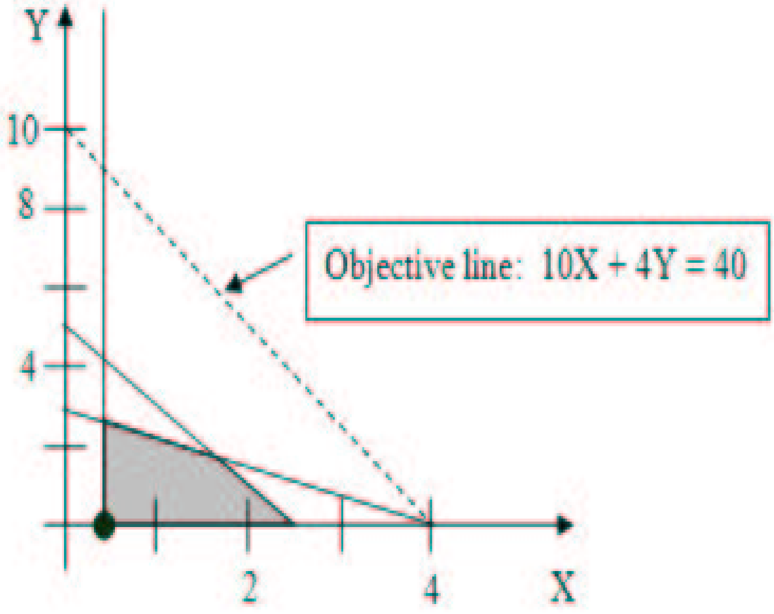

Figure 1: Figure for problem 10

9. (Ross 3.16, the lawyer contingency fee problem)

   (a)
   $$E[X] = \$5000$$

   (b)
   $$E[X] = 0.3 \times (25000) + 0.7 \times (0) = \$7500$$

   The contingency fee gives the lawyer the higher expected value, but that doesn't necessarily mean it is the best strategy. Perhaps a sure $5000 is better than $25000 with a probability of only 0.3. The decision is up to the lawyer.

10. (solving an LP with the graphical method)

    Graphical Solution:

    The feasible region is the shaded area on Figure 1. Since the objective line doesn't go through it, we must slide it parallel till it does. From there, since we are minimizing, we keep sliding down and to the left until we are about to leave the feasible region. If we do that, the objective will go through the black dot shown in the figure. This is thus the optimal point. Its coordinates are $(1, 0)$ by inspection. The optimal solution is thus $X = 1, Y = 0$. The optimal value of the objective will be $10 \cdot (1) + 4 \cdot (0) = 10$.

11. (Susie Sewingqueen problem)

    (a) Let

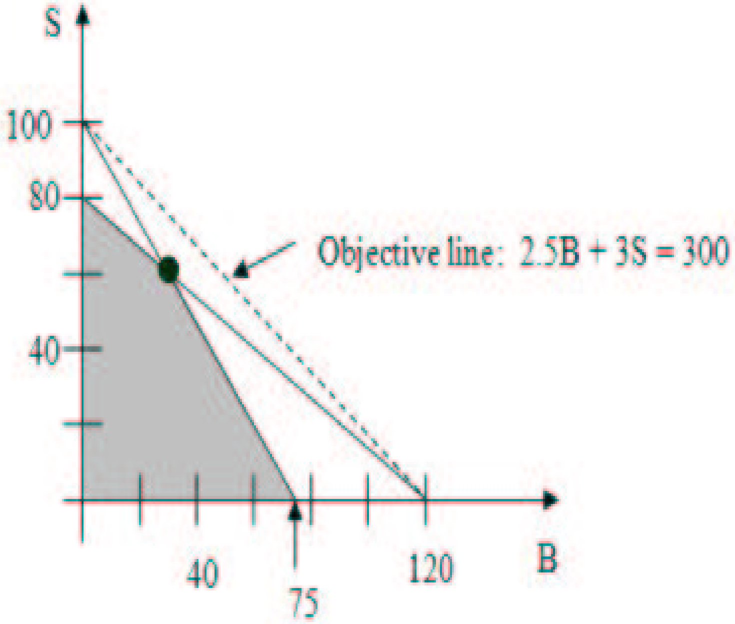

Figure 2: Figure for problem 11

$B = \#$Butterick patterns purchased
$S = \#$Simplicity patterns purchased

$$\begin{array}{rl} \text{maximize} & 2.5B + 3S \\ \text{subject to} & 6B + 9S \leq 720 \text{ (storage space)} \\ & 4B + 3S \leq 300 \text{ (spending limit)} \\ & B, S \geq 0 \text{ (non-negativity)} \end{array}$$

(b) Graphical Solution:
   Consider Figure 11. First, we must slide the objective line parallel to where it is shown above until it crosses through the feasible region. Then, we slide it in the optimal direction (up and to the right since this is a maximization problem) until we are about to leave the feasible region. When we do that, our objective line will go through the black dot shown in the figure. This is thus the optimal point. To find the optimal solution, we must find the (S, B) pair that satisfies both $6B + 9S = 720$ and $4B + 3S = 300$. If we do this, we obtain $B = 30$ and $S = 60$. Thus, Susie should stock 30 Butterick patterns and 60 Simplicity patterns. Her total profit will be $2.5 \cdot (30) + 3 \cdot (60) = \$255$.

   i. We must first add a new variable. Let $V = \#$Vogue patterns purchased. Add $3.25V$ to the objective function, $11V$ to the left-hand side of the storage space constraint, and $2.5V$ to the left-hand side of the spending limit constraint. Also, add $V \geq 0$.

ii. Add the constraint $S \geq V$.
iii. Add the constraint $V + B \leq 0.4 \cdot (V + S + B)$.

12. (the modified post office problem)

Denote the demand on day i by d_i.

First solution

- Variables:
  - x_i: number of employees who start work on day i, and work 5 consecutive days.
  - y_i: number of employees who start work on day i, and work 6 consecutive days.
- Objective:
  - Minimize $\sum_{i=1}^{7}(250x_i + (250+62)y_i)$.

To get the constraints right, we draw a table to indicate, on which days the employees are working.

day	1	2	3	4	5	6	7
employees x_1	x	x	x	x	x		
y_1	x	x	x	x	x	x	
x_2		x	x	x	x	x	
y_2		x	x	x	x	x	x
x_3			x	x	x	x	x
y_3	x		x	x	x	x	x
x_4	x			x	x	x	x
y_4	x	x		x	x	x	x
x_5	x	x			x	x	x
y_5	x	x	x		x	x	x
x_6	x	x	x			x	x
y_6	x	x	x	x		x	x
x_7	x	x	x	x			x
y_7	x	x	x	x	x		x

- So the constraints are, assuming that the demand for employees on day i is d_i:
  - $x_1 + y_1 + y_3 + x_4 + y_4 + x_5 + y_5 + x_6 + y_6 + x_7 + y_7 \geq d_1$.
  - $x_1 + y_1 + x_2 + y_2 + y_4 + x_5 + y_5 + x_6 + y_6 + x_7 + y_7 \geq d_2$.
  - etc.
  - $y_2 + x_3 + y_3 + x_4 + y_4 + x_5 + y_5 + x_6 + y_6 + x_7 + y_7 \geq d_7$.

Second solution

- Variables:

  - x_i: number of employees who start work on day i, and work 5 consecutive days.
  - y_i: number of employees who are forced to work overtime on day i, after completing their 5 day long schedule on day $i-1$.
- Objective:
  - Minimize $\sum_{i=1}^{7}(250x_i + 62y_i)$.

- The first group of constraints arise by simply adding y_i to the left hand side of the inequality in the original postoffice problem:
  - (*) $x_1 + x_4 + x_5 + x_6 + x_7 + y_1 \geq d_1$.
  - $x_1 + x_2 + x_5 + x_6 + x_7 + y_2 \geq d_2$.
  - etc.
  - (**) $x_3 + x_4 + x_5 + x_6 + x_7 + y_7 \geq d_7$.
- The second group of constraints force that the y_1 workers are drawn from the x_3 workers; the y_2 workers are drawn from the x_4 workers, etc.
  - $y_1 \leq x_3$; $y_2 \leq x_4$; $\ldots$; $y_7 \leq x_2$.